ORNL-CDC-1
UC-46 Criticality Studies

CRITICALITY OF LARGE SYSTEMS OF SUBCRITICAL U(93) COMPONENTS

J. T. Thomas

Neutron Physics Division
Oak Ridge National Laboratory

AUGUST 1967

OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee
operated by
UNION CARBIDE CORPORATION
for the
U. S. ATOMIC ENERGY COMMISSION
Contract No. W-7405-eng-26

FOREWORD

The Criticality Data Center has been established at the Oak Ridge National Laboratory under the auspices of the U. S. Atomic Energy Commission for the development of methods allowing extension and application of data derived from experiments and from analyses to problems in nuclear criticality safety, as well as for the review and evaluation of the data themselves. A necessary part of this program is a medium whereby information germane to the intent of the Center is made available. This report is the first in the series inaugurated for that purpose.

TABLE OF CONTENTS

CRITICALITY OF LARGE SYSTEMS OF SUBCRITICAL U(93) COMPONENTS

J. T. Thomas

ABSTRACT

Methods for estimating the number of components required for criticality of unreflected and paraffin reflected systems of subcritical units are described. A neutron nonleakage fraction parameter is defined and leads to a correlation confirmed to within 5% of the number of units by comparison with experimental data for three dimensional cuboidal arrays. A density analogue representation of the arrays is readily derivable and is shown to approximate the results from the above method, but is less precise. Factors by which the number of units in an unreflected critical array is reduced by adding a paraffin reflector are found to range from about six to greater than 30 depending on the material and on the average uranium density considered. The methods are supported by Monte Carlo calculations demonstrated to be reliable by comparison with the results of critical experiments.

INTRODUCTION

Present regulations[1] governing the storage and transport of fissile materials require the assessment of systems so large that experimental verification is often not practicable. This implies that judgement as to criticality - and hence, safety - is truly made on the credibility of the method of evaluation. It is not sufficient, therefore, only to check the method of estimation by experiments wherever possible, but there also should be an acceptable method for verifying extrapolations or criticality estimates that are far removed from experiments. The most likely manner to accomplish the latter is by means of detailed neutronics calculations such as are available in Monte Carlo techniques. A Monte Carlo search for criticality, however, can become an economic burden, especially if many areas are examined as in a parameter survey. It is feasible, however, to utilize a reliable Monte Carlo code to validate estimates of criticality made by semi-empirical methods which, in turn, are confirmed by the integral results of experiments.

In view of the large safety factors demanded by regulations in the specification of storage and transport, it seems reasonable that knowledge to within 5 or 10% of the number of components required to be critical at a given spacing is satisfactory.

The following presentation is intended to extend simple, well-known concepts applicable to individual critical units[2] to critical arrays of subcritical components. The methods and their limits of application are described, are applied to components utilized in critical experiments,[2-4] and are shown to yield estimates for configurations not studied experimentally. The estimates are validated by Monte Carlo calculations.

---

1. Regulations for the Safe Transport of Radioactive Materials, 1964 Revised Edition, International Atomic Energy Agency, Vienna (1965).
2. H. C. Paxton et al., "Critical Dimensions of Systems Containing U^{235}, Pu^{239}, and U^{233}," TID-7028, pp. 3-4 (June 1964).
3. J. T. Thomas, "Critical Three-Diemsnional Arrays of Neutron-Interacting Units," ORNL-TM-719 (October 1963).
4. J. T. Thomas, "Critical Three-Dimensional Arrays of Neutron-Interacting Units - Part II - U(93.2) Metal," ORNL-TM-868 (July 1964).

ANATOMY OF THE METHOD

We begin by noting that interest lies in what can be said about the criticality of large assemblages of subcritical units of fissile materials. We confine our attention to systems of identical units. This simplification assures us of a single characteristic average neutron energy spectrum independent of the number of units present. The fate of neutrons in a critical system is such that there is a balance established between the rates of absorption, production, and leakage. Since the first two must take place within the fissile material of an array, it is clear that an adjustment of leakage rate must be made to maintain criticality as the number of units is changed. Further, in order to maintain the balance between the three processes, the fraction of neutrons leaking must be independent of the number of pieces and the size of the system.

We assume, therefore, that we can approximate the array nonleakage fraction by a relation similar to $(1 + \bar{M}^2 B^2)^{-1}$, where B^2 is a geometric buckling and $\bar{M}^2$ a suitable neutron-energy-averaged migration area. We shall use the quantity NB_N^2 to replace $\bar{M}^2 B^2$ in the expression, where N is the number of units in a system and B_N^2 the corresponding system geometric buckling. The number of units in a critical system, N, is known to vary inversely with the average uranium density of the system to some power, not necessarily to the second power as theory would require. The density dependence is the only property of N that suggests its use as a coefficient of B_N^2. Further, the effects of non-uniformity, or lumping, of fissile material, unit size and unit shape variations on the usual nuclear parameters, which enter such a calculation, are not completely understood and we presume such changes are adequately approximated by the density exponent in some unknown way.

The correlations that follow are founded upon the two postulates

1. The fraction of neutrons that leak from different critical arrays of the same units is a constant, and
2. The nonleakage fraction is suitably represented by the expression $(1 + NB_N^2)^{-1}$.

Given two critical systems of identical units, one with N and the other with N' units, we may write

$$NB_N^2 = N'B_{N'}^2 \tag{1}$$

and since N and N' are arbitrary, it follows

$$NB_N^2 = \text{constant} \tag{2}$$

for arrays of the given unit. Equation (2) will, in all liklihood, subsist for any systematic method of computing the quantity B_N^2. The overall shape of the systems under consideration here will be, in general, cuboidal and we may, for convenience, adopt the usual simple expression for the buckling of this geometry:

$$B^2 = \sum_{i=1}^{3} \frac{\pi^2}{[a_i + 2\lambda]^2}$$

where a_i are the three dimensions of the system and λ is a suitable extrapolation distance. For systems of units we express a_i in terms of the unit dimensions, d_i, their surface separation, δ, and the number of units, n_i, along the ith direction, thusly,

$$B_N^2 = \sum_{i=1}^{3} \frac{\pi^2}{[n_i(d_i + \delta) + 2\lambda]^2} \tag{3}$$

where

$$N = \prod_{i=1}^{3} n_i \quad .$$

It is readily apparent that if the constant in Eq. (2) is known, the dependence of the critical spacing, δ, on the value of λ diminishes as N increases. Further, whenever experimental data exist for arrays of a given unit, it is possible to determine a suitable value of λ and of the constant in Eq. (2).

UNREFLECTED SYSTEMS

An effort to examine the behavior of Eq. (2) is made in Table 1 where the results of a number of three-dimensional, unreflected critical systems of fissile materials have been collected and the constants NB_N^2 and λ determined. For each case listed the value of λ satisfying Eq. (2) was sought. Considering the overall dimensions of the systems, the variation in λ is not large. There also appears to be a dependence of λ upon the units and on the array shape. For each entry in the Table the first column describes the critical assembly. The total number of units in an array is given as a subscript to the unit letter designation. The bracket, following the letter, contains the reflector thickness, zero in all cases; the unit surface separation; the corresponding average uranium density; and an array shape parameter, the ratio of the array height to the square root of its base area.

A description of an average unit in the array is contained in Columns 2-4. The number of units along the three edges of the array is given in Column 5. The value of λ and the constant NB_N^2 determined from Eq. (2) are given in Columns 6 and 7, respectively.

The last column of Table 1 contains the multiplication factor for each of the arrays computed by the GEM-3 Monte Carlo code.[5] These same systems were computed earlier using the GEM-1 code and the values of k were about 3% larger. The difference between the GEM-1 and GEM-3 results is due to a revision in the cross section sets used. The GEM code has been used to compute systems much more complex than those considered here. Its performance has been satisfactory[6] under such major variations as unit shape and mass, array shape, array reflection by hydrogeneous materials, the presence of moderator and container materials, two extreme spectra, mixed spectra, and the ^{235}U enrichment.

If we permit k values as determined by reliable Monte Carlo codes to be an acceptable substitute for experiment, then we have a credible means of evaluating systems not easily susceptible to experimental verification.

---

5. This code was furnished by the UKAEA Health and Safety Branch. See E. R. Woodcock et al., Session 6 of the International Conference on the Application of Computing Methods to Reactor Problems, ANL-7050 (May 1965).
6. J. T. Thomas, "Experimental and Calculated System Criticality," Table XV, Criticality Control of Fissile Materials, IAEA, Vienna (1966).

Table 1. Unreflected Critical Arrays of Uranium Metal and Solution

Array Description[a]	Unit Description[c] Av Mass (kg U)	Diam (cm)	Height (cm)	N $n_1n_2n_3$	λ cm [From Eq.(2)]	NB_N^2	k_{eff} (GEM)[d]
			U(93.2) Metal Units				
A_{27}^2 {0;2.007;7.767;0.55}	10.484	11.509	5.382	333	1.86	0.670	1.0022
A_{64}^6 {0;3.952;4.693;0.61}	10.434	11.481	5.382	444	1.86	0.670	0.9937
A_{27}^4 {0;2.436;7.096;0.96}	10.489	9.116	8.641	333	1.58	0.573	0.9919
A_{64}^4 {0;4.625;4.187;0.96}[b]	10.487	9.116	8.641	444	1.58	0.573	1.0006
B_8^1 {0;0.902;11.374;0.73}	15.692	11.494	8.077	222	1.63	0.376	1.0349
B_{27}^2 {0;4.204;5.185;0.78}	15.683	11.490	8.077	333	1.63	0.376	0.9836
C_8^2 {0;2.248;8.514;0.95}	20.960	11.506	10.765	222	1.92	0.249	1.0047
C_{27}^3 {0;6.363;3.827;0.96}	20.877	11.484	10.765	333	1.92	0.249	0.9961
D_8^1 {0;3.543;6.806;1.18}	26.218	11.509	13.459	222	2.56	0.179	0.9894
D_{27}^2 {0;8.484;2.980;1.10}	26.113	11.486	13.459	333	2.56	0.179	0.9955
			5-liter $U(92.6)C_2(NO_3)_2$ Solution Units				
F_8^1 {0;1.43;0.214;0.94}	2.074	20.32	19.05	222	0.510	0.124	1.0166
F_{27}^1 {0;6.48;0.114;0.95}	2.074	20.32	19.05	333	0.510	0.124	0.9979
F_{64}^1 {0;10.67;0.072;0.96}	2.074	20.32	19.05	444	0.510	0.124	0.9926
F_{125}^1 {0;14.40;0.052;0.96}	2.074	20.32	19.05	555	0.510	0.124	1.0011
F_8^2 {0;1.43;0.144;0.94}	1.395	20.32	19.05	222	0.335	0.126	0.9944
F_{27}^2 {0;6.40;0.076;0.95}	1.395	20.32	19.05	333	0.335	0.126	1.0017

a. Array description follows notation in Ref. 4, briefly,

$$X \text{ unit designation } \left\{\begin{matrix}\text{Refl.} \\ \text{Thickness}\end{matrix}\right. ; \text{ Surface Separation}; \text{ Av U Density}; \left\{\frac{\text{Array ht}}{\sqrt{\text{base area}}}\right\}$$
$$\text{No. units in array}$$

b. Estimated spacings.

c. Uranium density of metal units is 18.76 g/cm^3.

Solution Units	g U/ℓ	Sp. gravity	Atomic Ratio H:^{235}U
F^1	415	1.555	59
F^2	279	1.373	92

d. GEM-3 Monte Carlo code was used to determine multiplication factor.

Systems of Large N

Equation (2) and the values of λ appearing in Table 1 permit the critical conditions for systems of interest to nuclear safety specialists to be estimated. Presented in Table 2 are the results of applying Eq. (2) to the units of Table 1. The arrays are three dimensional, cuboidal in shape, with an equal number of units along each of the three dimensions. The given k values are the result of subjecting each of the estimated spacings to a Monte Carlo calculation.

The multiplication factors appearing in Table 2 are comparable to values given in Table 1 obtained for the experiments with these units. We also note that the array shape tends toward the optimum value of unity independent of the unit shape as N increases. This supports an earlier experimental observation[4] that slight changes in unit shape result in smaller reactivity contributions to arrays than do slight changes in array shape.

The data of Tables 1 and 2 are presented graphically in Figs. 1 and 2. The metal systems shown in Fig. 1 display the effect of unit size on the critical number as a systematic decrease in average uranium density for increases in mass of a unit or more optimum unit shape. The slight variations in the slope of the lines for large N is similar to the limits observed by Abbey[7] from Monte Carlo calculations on spherical units. He noted that for $N > 64$ the values of the slope as a function of the spherical mass fell within the range 1.73 and 1.86. DTK calculations reported by Paxton[8] give a slope of 1.8 for the solution systems shown in Fig. 2.

The direct application of Eq. (2) to other than the existing experimental units is not possible. If the constant, NB_N^2, is not known for an arbitrary unit, it may be possible to interpolate from the experimentally determined values in the case of a unit which is comparable to those used in experiments. One may also utilize Monte Carlo calculations to establish two "critical" configurations and then apply Eq. (2). It is stressed,

---

7. F. Abbey, "The Criticality of Interacting Arrays," Criticality Control of Fissile Materials, IAEA, Vienna (1966).
8. H. C. Paxton, "Criticality Control in Operations with Fission Material," LA-3366, Los Alamos Scientific Laboratory (January 1966).

Table 2. Critical Arrays Estimated by the NB_N^2 Method and Computed Multiplication Factors.

Array Description[a]		k_{eff} (GEM)	Array Description[a]		k_{eff} (GEM)
A^2_{125}	{0;5.599;3.257;0.64}	--	C^3_{64}	{0;9.602;2.306;0.97}	0.9881
A^2_{216}	{0;7.060;2.439;0.67}	0.9875	C^3_{125}	{0;12.365;1.587;0.97}	--
A^2_{512}	{0;9.619;1.562;0.71}	0.9978	C^3_{216}	{0;14.818;1.180;0.97}	0.9937
A^2_{1000}	{0;11.852;1.112;0.74}	0.9916	C^3_{512}	{0;19.107;0.747;0.98}	1.0023
A^2_{8000}	{0;20.552;0.3921;0.81}	--	C^3_{1000}	{0;22.841;0.527;0.98}	0.9914
			C^3_{8000}	{0;37.307;0.182;0.99}	--
A^4_{64}	{0;4.625;4.187;0.97}	1.0006			
A^4_{125}	{0;6.479;2.852;0.97}	--	D^2_{64}	{0;12.363;1.778;1.08}	1.0022
A^4_{216}	{0;8.118;2.107;0.97}	0.9767	D^2_{125}	{0;15.653;1.218;1.07}	--
A^4_{512}	{0;10.973;1.325;0.98}	1.0074	D^2_{216}	{0;18.567;0.903;1.07}	0.9982
A^4_{1000}	{0;13.451;0.932;0.98}	0.9758	D^2_{512}	{0;23.653;0.570;1.06}	0.9853
A^4_{8000}	{0;23.020;0.321;0.99}	--	D^2_{1000}	{0;28.074;0.402;1.05}	0.9800
			D^2_{8000}	{0;45.181;0.139;1.03}	--
B^2_{64}	{0;6.815;3.143;0.81}	0.9983			
B^2_{125}	{0;9.048;2.171;0.83}	--	F^1_{216}	{0;17.662;0.0392;0.97}	0.9802
B^2_{216}	{0;11.032;1.618;0.85}	1.0084	F^1_{512}	{0;23.531;0.0253;0.97}	0.9791
B^2_{512}	{0;14.505;1.028;0.87}	1.0005	F^1_{1000}	{0;28.691;0.0181;0.97}	1.0066
B^2_{1000}	{0;17.531;0.727;0.88}	1.0202	F^1_{8000}	{0;48.893;0.0064;0.98}	--
B^2_{8000}	{0;29.276;0.253;0.92}	--			

Table 2. (Cont'd)

Array Description[a]	k_{eff} (GEM)
F^2_{64} {0;10.553;0.0494;0.96}	1.0032
F^2_{216} {0;17.482;0.0267;0.97}	0.9755
F^2_{512} {0;23.306;0.0173;0.97}	0.9832
F^2_{1000} {0;28.429;0.0124;0.97}	0.9921
F^2_{2000} {0;48.500;0.00436;0.98}	--

a. $X^{\text{Unit design}}_{\text{No. Units in Array}}$ { Reflector thickness (cm); Surface Separation (cm); Av Uranium Density (g/cm^3); $\frac{\text{Array Height}}{\sqrt{\text{base area}}}$ }

See Table 1 for unit description.

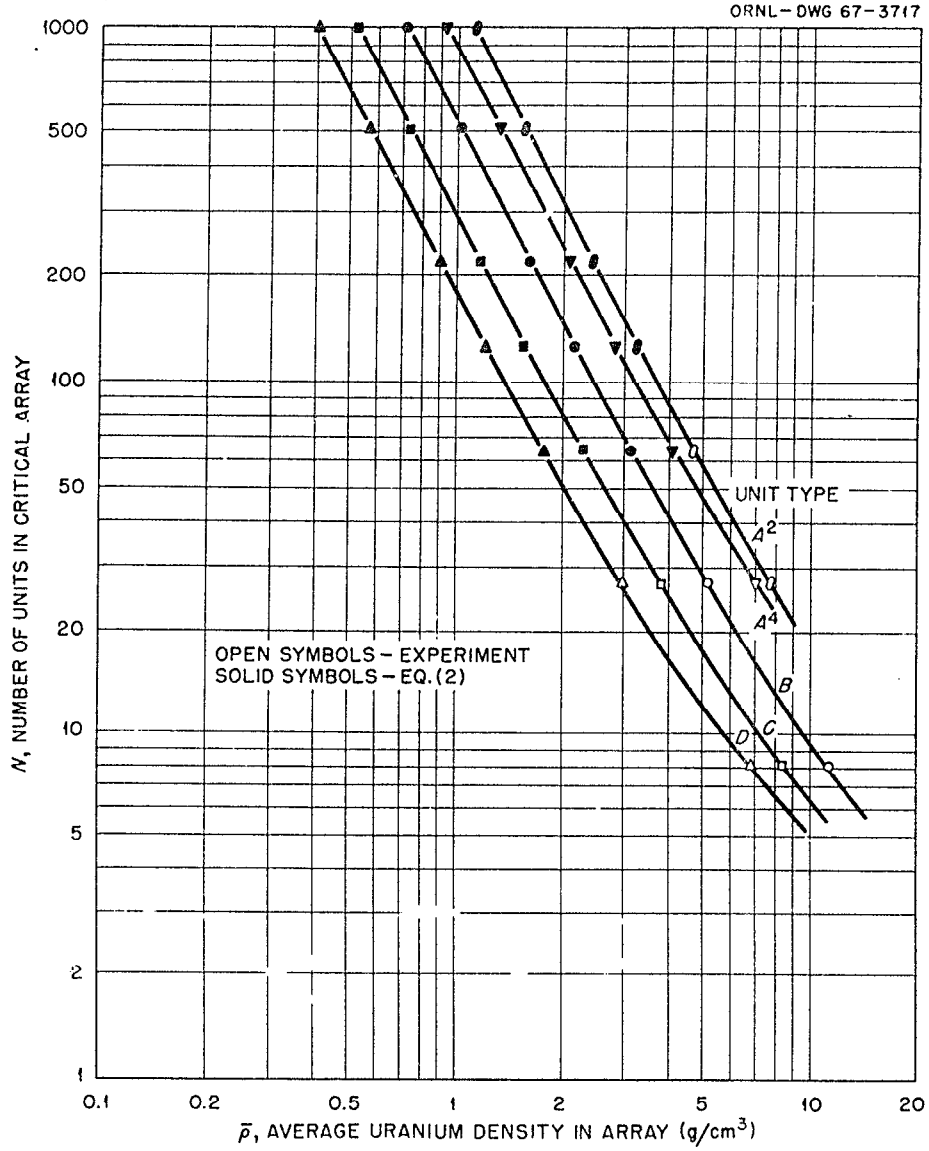

Fig. 1. Number of Units of U(93.2) Metal in Critical Arrays as a Function of the Average Uranium Density. (Experimental data of Table 1 and computed data of Table 2 are included.)

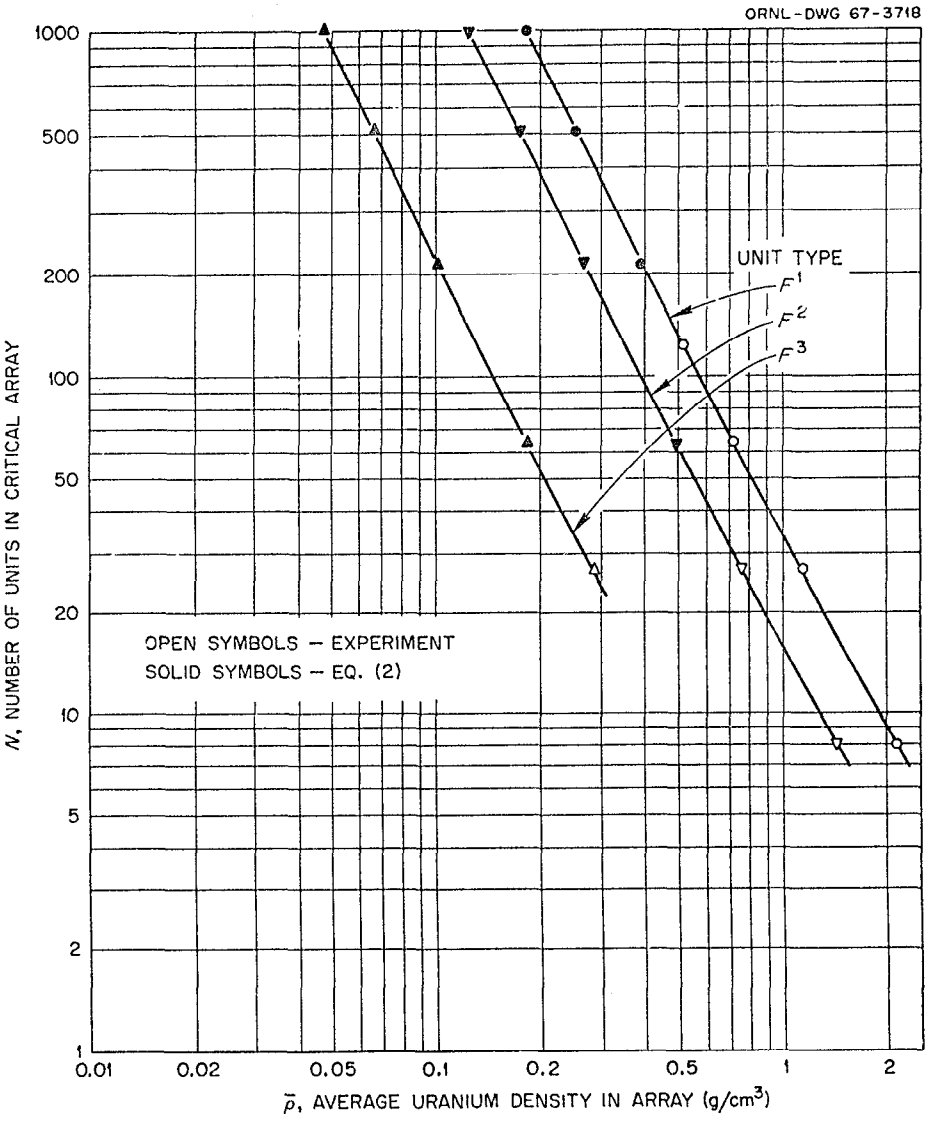

Fig. 2. Number Units of $U(92.6)O_2(NO_3)_2$ Aqueous Solution in Critical Arrays as a Function of the Average Uranium Density. (Experimental data of Table 1 and computed data of Table 2 are included.)

however, that one should demonstrate the ability of any proposed code to compute critical experiments with the same material composition as the units under question.

At the expense of the accuracy possible in the use of Eq. (2), we derive in the following section a method which is an approximation to the results of Eq. (2) over a range of interest to criticality safety.

Density Analogue Representation

Let us accept the results of Eq. (2) for large N as an extension of the experimental data listed in Table 1. Let α be an index running through the units X^i of Table 1 and define the quantity $\underline{\bar{K}}(\alpha)$ as the fraction of neutrons leaking from an N-unit critical array of α units, thus,

$$\underline{\bar{K}}(\alpha) = \frac{NB_N^2}{1 + NB_N^2} \quad . \tag{4}$$

We begin by noting a productive correlation of experimental data by means of this leakage parameter. Since the leakage fraction from every array of a particular α unit is assumed constant, it must be independent of the average uranium density of the system. We are led to ask, therefore, what is the relation between $\underline{\bar{K}}(\alpha)$ and $\bar{\rho}$ (the unit mass divided by the cell volume the unit occupies in the array) as a function of α. The results for the cylindrical metal units are displayed in Fig. 3 where the leakage is shown as a function of the average uranium density for the different units.

With the exception of the eight unit arrays, the data for different α but equal N appear to enjoy a linear relation including the point at the origin. We would like to relate the three quantities N, $\underline{\bar{K}}(\alpha)$, and $\bar{\rho}$. This is most readily accomplished through the slope of the lines. Let θ be the angle made by the line for any given N with the abscissa; then, expressed as a least squares fit

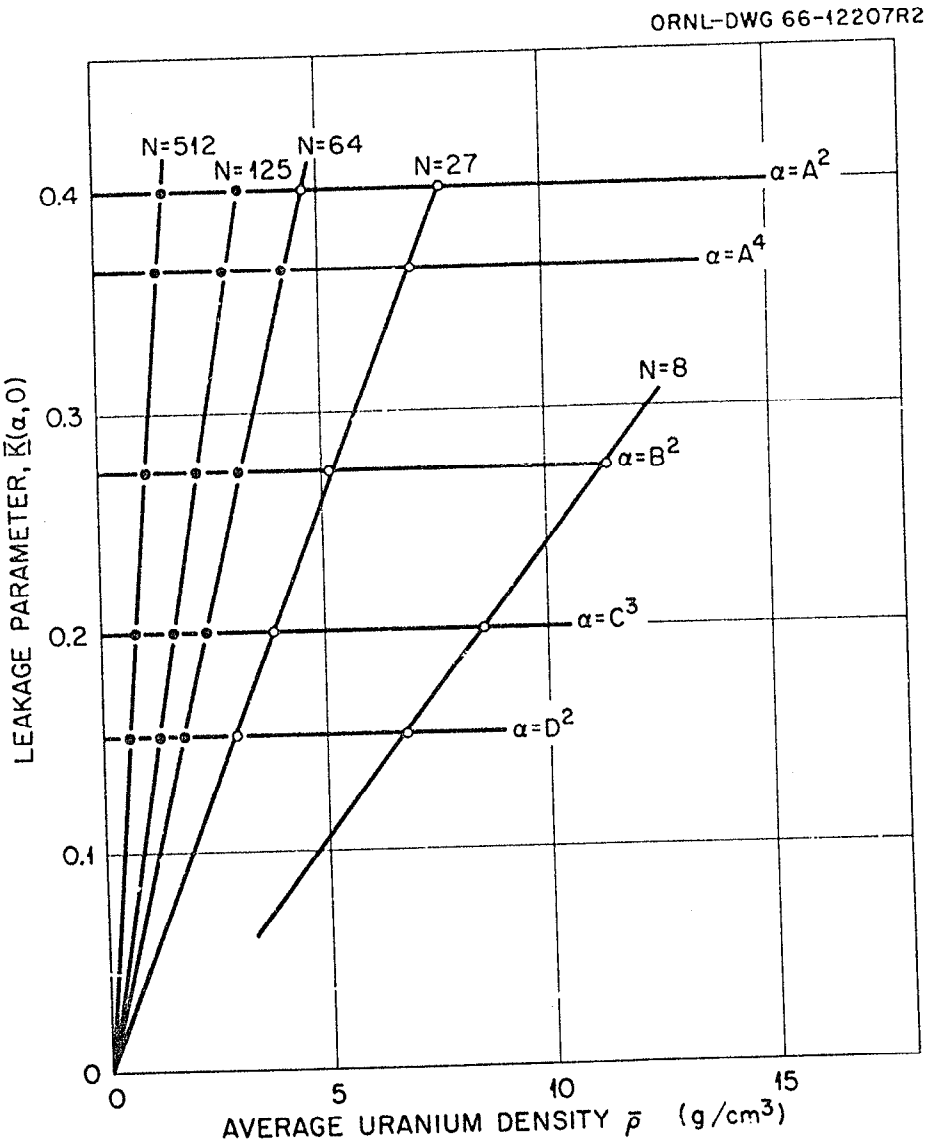

Fig. 3. The Leakage Parameter as a Function of the Average Uranium Density of Unreflected Critical Arrays of Cylindrical U(93) Metal Units.

$$\frac{\sum_{\alpha} \bar{\underline{K}}(\alpha)}{\sum_{\alpha} \bar{\rho}(\alpha)} = \tan\theta \quad . \tag{5}$$

If we next plot N as a function of $\tan\theta$, we find the following relation subsists for $N \geq 27$:

$$N = A\ (\tan\theta)^{s} \tag{6}$$

where A and s are constants independent of α.

Combining Eqs. (5) and (6) we obtain an equation in the form of the density analogue representation

$$N = A \left[\frac{\bar{\underline{K}}(\alpha)}{\bar{\rho}(\alpha)} \right]^{s} \quad . \tag{7}$$

The constants A and s of Eq. (6) or (7) are presented in Table 3 for the metal cylinders and for each of the solution units described in Table 1. Unlike the results for the uranium-metal cylinders, where the ^{235}U concentration in the unit is constant and the unit size is changed, the two sets of constants F^1 and F^2 characterize materials having two separate and distinct spectra and, hence, the data for these different units cannot be represented by a single equation. It is expected, however, that each equation defined by the constants would adequately represent the criticality of different sized units at the same concentration.

It is emphasized that Eq. (7) is an approximation to the "experimental data" represented by Eq. (2). The distinction is clearly demonstrated in Fig. 4 where the results for the B and C units are presented from both Eq. (2) and Eq. (7). A further comparison is made by examining the ratio of the critical density from Eq. (7) to that determined from Eq.(2) for a given N; this is presented in Table 4. The approximation may be considered good in the range from 27 to 8000 units. Outside this range, i.e., for N less than 27 or greater than 8000 units, Eq. (7) will yield densities that may be considered conservative.

Table 3. The Constants of Eqs. (6) and (7) for the Various Units of Table 1.

Units	A $(g/cm^3)^s$	s
Metal Cylinders	6410.2	1.880
F^1	27.804	1.974
F^2	12.558	1.985

Table 4. Ratio of Estimated Critical Density $\bar{\rho}$ from Eq. (7) to that from Eq. (2) for Cylindrical U(93.2) Metal Units and $U(92.6)O_2(NO_3)_2$ Solution Units

	Type of Unit, α						
N	A^2	A^4	B	C	D	F^1	F^2
27	0.9503	0.9424	0.9669	0.9567	0.9338	0.9959	0.9946
64	0.9907	1.0091	1.0080	1.0034	0.9888	1.0025	0.9998
125	0.9999	1.0376	1.0219	1.0211	1.0111	1.0046	1.0018
216	0.9981	1.0500	1.0251	1.0269	1.0197	1.0048	1.0024
349	0.9922	1.0545	1.0234	1.0272	1.0218	1.0043	1.0024
512	0.9847	1.0550	1.0196	1.0249	1.0207	1.0035	1.0021
10^3	0.9690	1.0502	1.0094	1.0167	1.0141	1.0013	1.0011
$8x10^3$	0.9091	1.0098	0.9611	0.9722	0.9719	0.9913	0.9958

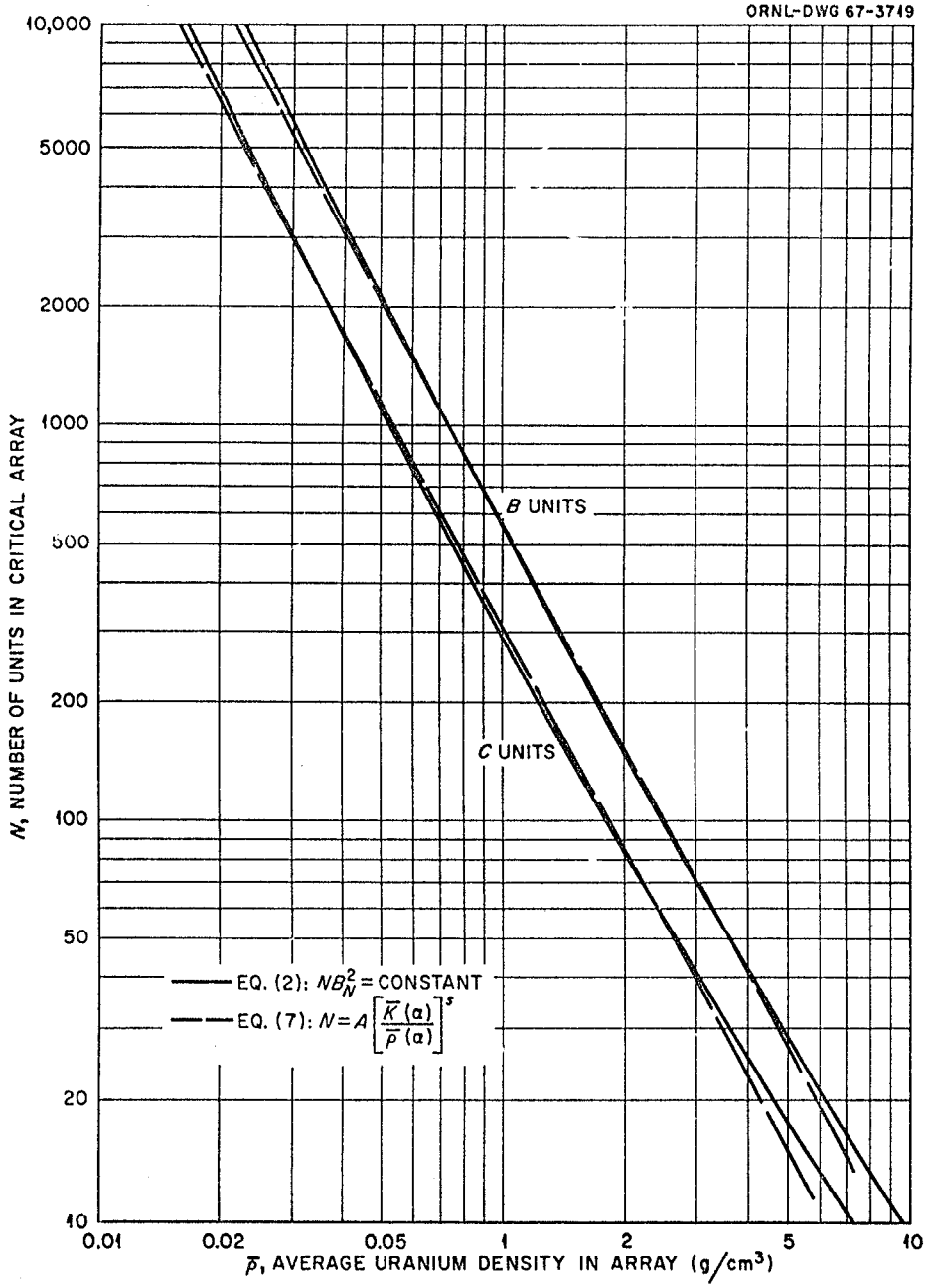

Fig. 4. Comparison of Predictions of the Dimensions of Critical Arrays by Eqs. (2) and (7).

Thus far, we have concerned ourselves with the experimental units. It would be desirable to complete the generalization of Eq. (7) for metal units by establishing a relation between $\underline{\bar{K}}(\alpha)$ and arbitrary cylindrical metal units. This can be accomplished in a number of ways, but the most useful, perhaps, is an expression of $\underline{\bar{K}}(\alpha)$ as a function of the surface-to-volume ratio of the unit. Listed in Table 5 are the S/V ratio, the unit k_{eff}, the fraction, $f(\alpha)$, of neutrons leaking from the unit, and the $\underline{\bar{K}}(\alpha)$ values determined for the experimental units α. The following relations subsist for these data:

$$f(\alpha) = 1 - 0.177\ [S/V]^{-0.959} \tag{8}$$

and

$$\frac{\underline{\bar{K}}(\alpha)}{f(\alpha)} = 1.387\ \frac{S}{V} - 0.450\ . \tag{9}$$

The maximum percent error* in $\underline{\bar{K}}(\alpha)$, using Eqs. (8) and (9) for the experimental units, is 4.6; from Eq. (7), this produces an error in N of $\sim 9\%$.

Equations (7), (8), and (9) allow one to estimate the critical spacing for a given metal unit simply from its surface-to-volume ratio. This set of equations is valid for systems which are characterized by:

i. U(93) cylindrical metal units,
ii. Three dimensional cuboidal arrays of identical units,
iii. Unreflected and unmoderated systems, and
iv. $27 \leq N \leq 8000$.

The application of Eqs. (2), (7), (8), and (9) is exemplified by estimating the critical densities and spacings for three arrays of specified units when an N value of 216 is assumed. The $\underline{\bar{K}}(\alpha)$ values as determined from Eqs. (8) and (9) give the constants to be used with Eqs. (2) and (7). The results are presented in Table 6; the values of λ were estimates interpolated from values appearing in Table 1 since their influence is small and a slide rule was used throughout. The systems derived from

*The formulas are intended for "slide rule" accuracy.

Table 5. Some Properties of the Cylindrical Metal Units.

Unit,[a] α	S/V, cm^{-1}	k_{eff} [b]	Neutron Leakage Fraction,[b] $f(\alpha)$	$\bar{K}(\alpha)$
A^2	0.719	0.566	0.756	0.401
A^4	0.670	0.595	0.744	0.364
B^2	0.596	0.685	0.707	0.273
C^3	0.534	0.759	0.673	0.200
D^2	0.497	0.799	0.656	0.152

a. See Table 1 for physical description of units.

b. Determined by Monte Carlo calculations.

Table 6. Comparison of $\bar{\rho}$ and δ as Determined by Eqs. (2) and (7) for Non-experimental Systems of 216 Units.

Unit Description				Eq. (2)				Eq. (7)	
α	Mass, kg U	Diam cm	Height cm	Est. λ cm	$\bar{\rho}$ g U/cm^3	δ cm	Monte Carlo k_{eff}	$\bar{\rho}$ g U/cm^3	δ cm
E^4	5.0	7.094	6.739	0.5	3.89	3.90	1.0060	3.63	4.15
B^5	15.68	10.388	9.869	1.7	1.48	11.73	0.9750	1.58	11.27
C^6	21.01	9.116	17.282	1.8	1.35	13.40	0.9971	1.35	13.39

Eq(2) were calculated by the GEM Monte Carlo code and the resulting multiplication factors are reported in the table. The results of Eq. (2) appear reasonable while those of Eq. (7), which approximate the former, underestimate the critical density for the 5 kg unit, overestimate that for the 15 kg unit and agree in the case of the 21 kg unit. In summary, then, the results of Eq. (7) are within 12% of those from Eq. (2) for these units. The results for the A^2 unit appearing in Table 4 and those for the 5 kg unit of Table 6 suggest that as metal units become smaller the value of the slope used in Eq. (7) becomes larger,* approaching a value near 2. The range of values for the slope, 1.88 to 2.0, however, is not significant for the purposes to which Eq. (7) should be applied, and a value of 1.88 will be conservative for small unit masses.

PARAFFIN REFLECTED SYSTEMS

The experimental investigation of critical reflected arrays beyond 64 components has not been possible because of limitations imposed by equipment and material requirements. The need for information concerning larger values of N in the low uranium density range becomes manifest when it is realized that factors for reflector addition to an array, as used in some safety specifications, are those determined in the experimental density range. Although it may be correct to assume the magnitude of the factor approaches a constant value as the density decreases, it is not clear from the present evidences of the reflector effectiveness that this will occur for other than extremely low average densities.

It is the intent here to apply the postulates of page 3 and Eqs. (2) and (7) to the available experimental data on three dimensional cuboidal systems. Summarized in Table 7 are the experimental data[4] for U(93.2) cylindrical metal units. Table 8 gives a summary of the data[3] for the 5-liter units of uranyl nitrate solution. The units are those described

*This effect has been confirmed over larger ranges of N for the individual and collective units.

Table 7. Critical Conditions for Regular Three Dimensional Arrays with Various Paraffin Reflectors

Array Description[a]	Paraffin Reflector Thickness (cm)	Surface Separation[b] of Units (cm)	Average Uranium Density in Array (g/cm^3)	Ratio of Array Height to $\sqrt{\text{Base Area}}$
A_8^1 (2x2x2)	0	0[c]	14.709	0.47
	1.3	0.229	13.563	0.48
	3.8	1.981	7.825	0.55
	7.6	3.416	5.350	0.59
	15.2	3.696	4.995	0.60
A_{27}^2 (3x3x3)	0	2.007	7.767	0.55
	1.3	2.992	5.954	0.58
	3.8	5.872	3.085	0.65
	7.6	8.258	1.967	0.69
	15.2	8.689	1.826	0.70
A_8^3 (2x2x2)	0	0[d]	14.632	0.95
	1.3	0.602	12.037	0.95
	3.8	2.362	7.248	0.96
	7.6	3.970	4.865	0.96
	15.2	4.308	4.503	0.97
A_{27}^4 (3x3x3)	0	2.436	7.096	0.96
	1.3	3.426	5.526	0.96
	3.8	6.579	2.798	0.97
	7.6	9.017	1.807	0.97
	15.2	9.434	1.686	0.97
A_{64}^6 (4x4x4)	0	3.952	4.693	0.61
	15.2	12.360	1.035	0.74
B_8^1 (2x2x2)	0	0.902	11.374	0.73
	1.3	1.905	8.756	0.75
	3.8	4.961	4.445	0.79
	7.6	7.391	2.845	0.82
	15.2	7.823	2.645	0.82
B_{27}^2 (3x3x3)	0	4.204	5.185	0.78
	1.3	5.677	3.869	0.80
	3.8	10.190	1.827	0.84
	7.6	13.693	1.137	0.86
	15.2	14.194	1.067	0.87

Table 7. Continued

Array Description[a]	Paraffin Reflector Thickness (cm)	Surface Separation[b] of Units (cm)	Average Uranium Density in Array (g/cm^3)	Ratio of Array Height to $\sqrt{\text{Base Area}}$
C_8^1 (2x2x2)	0	2.217	8.562	0.95
C_8^2 (2x2x2)	0	2.248	8.514	0.95
	1.3	3.678	6.295	0.95
	2.5	5.710	4.292	0.96
	3.8	8.207	2.843	0.96
	7.6	11.509	1.777	0.97
	15.2	11.986	1.669	0.97
C_{27}^3 (3x3x3)	0	6.363	3.827	0.96
	1.3	8.574	2.683	0.96
	3.8	14.764	1.187	0.97
	7.6	18.720	0.776	0.98
	15.2	19.147	0.744	0.98
D_8^1 (2x2x2)	0	3.543	6.806	1.18
	1.3	5.423	4.843	1.12
	3.8	11.532	1.976	1.09
	7.6	15.697	1.215	1.07
	15.2	16.378	1.130	1.07
D_{27}^2 (3x3x3)	0	8.494	2.980	1.10
	1.3	11.323	2.025	1.09
	3.8	19.606	0.817	1.06
	7.6	24.498	0.531	1.05
	15.2	24.991	0.510	1.05

a. The letter and the superscript identify the average unit in the array described in Table 1; the subscript is the number of units in the array; the numbers in parentheses are the horizontal and vertical dimensions, respectively, of the array expressed in number of units.

b. Errors on all surface separations are $\pm$ 0.013 cm for unreflected arrays and $\pm$ 0.026 cm for reflected arrays.

c. Array was subcritical with an apparent neutron source multiplication of ~ 3.

d. Array was subcritical with an apparent neutron source multiplication of ~ 10.

Table 8. Critical Conditions for Regular Three Dimensional Arrays of $U(92.6)O_2(NO_3)_2$-Five Liter Solution Units with Various Paraffin Reflectors

Array Description[a]	Paraffin Reflector Thickness (cm)	Surface Separation of units[b] (cm)	Average Uranium Density in Array (g/cm^3)	Ratio of Array Height to $\sqrt{\text{Base Area}}$
F_8^1 (2x2x2)	0	1.43	0.214	0.94
	1.3	3.28	0.167	0.95
	3.8	6.91	0.108	0.95
	7.6	8.48	0.091	0.96
	15.2	8.99	0.087	0.96
F_{27}^1 (3x3x3)	0	6.48	0.114	0.95
	1.3	9.02	0.086	0.96
	15.2	16.53[c]	0.043	0.96
F_{64}^1 (4x4x4)	0	10.67	0.072	0.96
F_{125}^1 (5x5x5)	0	14.40	0.052	0.96
F_8^2 (2x2x2)	0	1.43	0.144	0.94
	11.4[d]	8.71	0.060	0.96
F_{27}^2 (3x3x3)	0	6.40	0.077	0.95

a. The letter and the superscript identify the average unit in the array described in Table 1; the subscript is the number of units in the array; the numbers in parentheses are the horizontal and vertical dimensions, respectively, of the array expressed in number of units.

b. The uncertainty in the values of the separation is $\pm$ 0.13 cm.

c. The separation was 16.91 cm where one face of the array was reflected by Plexiglas 15.2-cm-thick.

d. The array was reflected on the bottom by 15.2-cm-thick paraffin.

in Table 1. The ability of the Monte Carlo codes to compute these systems has been reported in Ref. 6. We shall continue to use such calculations as a valuation of the methods when applied to reflected systems.

Clarification of the application and results of the equations will be made by introducing, where appropriate, the parameter R for the reflector thickness about an array.

As in the case for unreflected systems, we use the experimental data for two critical systems with the same units to determine the constants λ and NB_N^2 of Eq. (2). A few representative determinations of the systems of Tables 7 and 8 for R = 15.2 cm are given in Table 9 where the calculated multiplication factors are also shown. Comparison of the λ-values with those of Table 1 for unreflected systems shows that larger values of λ are necessary when a reflector is present as is the case for individual critical assemblies.[2] Values of λ for the intermediate reflector thicknesses lie between those for the two reflector conditions given in Tables 1 and 9.

Reflected Systems of Large N

We proceed to extend the data of Tables 7 and 8 to values of N greater than 27 be means of Eq. (2) and the constants of Table 9. Estimated critical conditions for the various units of Table 1 are presented in Table 10 along with a multiplication factor computed for a representative array from each group. With the exception of the B unit, the multiplication factors are comparable to those computed for the experiments of Table 9. We will accept the results for the B units as given.

The application of the postulates to the reflected systems of the experimental units is seen to yield reasonable results and, for nuclear safety purposes, we assume Eq. (2) adequately represents the criticality of large reflected systems. Considering the computed arrays as an extension of the experimental data, we proceed to derive a density analogue representation for thick-reflected systems as was done for the unreflected systems.

Table 9. Representative Values of the Constants λ and NB_N^2 for U(93) Metal and Solution Components in Thick Paraffin Reflected Systems.

Array Description[a]	λ cm	NB_N^2	k[b]
A_{64}^{6} {15.2;12.360;1.035;0.74}	4.08	0.179	0.9815
A_{8}^{3} {15.2;4.308;4.503;0.97}	3.90	0.168	1.0021
B_{8}^{1} {15.2;7.823;2.645;0.82}	4.23	0.107	0.9869
C_{27}^{3} {15.2;19.147;0.744;0.98}	3.45	0.0831	1.027*
D_{27}^{2} {15.2;24.991;0.510;1.05}	3.94	0.0563	1.028*
F_{8}^{1} {15.2;8.99;0.087;0.96}	1.89	0.0589	1.021*
F_{27}^{1} {15.2;16.53;0.043;0.97}	1.89	0.0589	1.0056

a. See Table 1 for an explanation of array description.

b. The starred values are computed by GEM-1 and the unstarred values by GEM-3.

Table 10. Critical Arrays Estimated by the NB_N^2 Method and Representative Computed Multiplication Factors for the Units of Table 1.

Array Description[a]	k_{eff}	Array Description[a]	k_{eff}
A^2_{216} {15.2;18.32;0.496;0.81}	--	C^3_{216} {15.2;33.83;0.227;0.98}	--
A^2_{512} {15.2;23.06;0.308;0.84}	0.9980	C^3_{512} {15.2;41.27;0.144;0.99}	0.9916
A^2_{1000} {15.2;27.14;0.215;0.85}	--	C^3_{1000} {15.2;47.74;0.101;0.99}	--
A^4_{216} {15.2;19.39;0.460;0.98}	--	D^2_{216} {15.2;42.75;0.157;1.04}	--
A^4_{512} {15.2;24.29;0.285;0.99}	1.0033	D^2_{512} {15.2;51.76;0.100;1.03}	1.0057
A^4_{1000} {15.2;28.52;0.199;0.99}	--	D^2_{1000} {15.2;59.62;0.071;1.03}	--
B^2_{216} {15.2;26.83;0.305;0.91}	--	F^1_{216} {15.2;33.76;0.014;0.98}	--
B^2_{512} {15.2;33.12;0.191;0.92}	0.9507	F^1_{512} {15.2;41.15;0.0091;0.98}	1.0251
B^2_{1000} {15.2;38.57;0.134;0.93}	--	F^1_{1000} {15.2;48.51;0.0064;0.98}	--

a. See Table 1 for array description.

Density Analogue Representation With Reflector Present

Introducing the parameter R into Eq. (7) to represent the thickness of reflector about an array we obtain

$$N(R) = A(R)\left[\frac{\underline{\bar{K}}(\alpha,R)}{\bar{\rho}(\alpha,R)}\right]^{s(R)} \qquad (10)$$

where Eq. (7) is now a special case for R = 0. The values of $\underline{\bar{K}}(\alpha,R)$ as determined for the various reflector thicknesses and units of Tables 7 and 8 are given in Table 11. $\underline{\bar{K}}(\alpha,R)$, $\bar{\rho}(\alpha,R)$ and Eqs. (5) and (6) provide the constants A(R) and s(R) given in Table 12. An index of the accuracy of Eq. (10) in representing the results of Eq. (2) is the ratio of the average density given by Eq. (10) to that given by Eq. (2). This comparison is made in Table 13 for an R value of 15.2 cm, and it may be seen that the critical densities are within a few percent of each other except for the A^4 unit where the maximum difference is about 7 percent. An error of $\pm$ 2% in $\bar{\rho}$ corresponds to an error in N of $\pm$ 3.6%. We are ignoring the N = 27 and 8000 values since the straight line approximation to Eq. (2) should conservatively underestimate the critical densities for these values. If we were to not use the group averaged values of A(R) and s(R) for the metal cylinders, but rather their individual values, then the error in the approximation would be comparable to that for the F^1 units.

Factors for Reflection

Often it is convenient to know the ratio of the number of units in an unreflected critical array to that in a reflected critical array of the same average uranium density. The ratio is sometimes referred to as the reflection factor and is conveniently used when it is possible to more reliably estimate the critical conditions for an unreflected array. We shall show that such a factor is dependent on the unit size, density of fissionable material (or neutron energy spectrum), average density of the array, and the degree of array reflection.

Table 11. Neutron Leakage Fraction for Reflected and Unreflected Critical Arrays of the Units in Table 1.

Unit α	$\bar{K}(\alpha,0)$	$\bar{K}(\alpha,R^*)$			
		$\bar{K}(\alpha,1.3)$	$\bar{K}(\alpha,3.8)$	$\bar{K}(\alpha,7.6)$	$\bar{K}(\alpha,15.2)$
A^2	0.401	0.355	0.244	0.185	0.179
A^4	0.364	0.338	0.224	0.174	0.168
B^2	0.273	0.232	0.148	0.110	0.107
C^3	0.200	0.159	0.0980	0.0778	0.0768
D^2	0.152	0.118	0.0676	0.0535	0.0533
F^1	0.111	0.0924	0.0695	--	0.0589

*Reflector thickness in cm.

Table 12. Values for the Constants A(R) and s(R) of Eq. (10) for the Units of Table 1 as a Function of the Reflector Thickness, R.

Units, α	R, cm	A(R)	s(R)
A^2, A^4, B^2, C^3, or D^2	0	6410.1	1.880
	1.3	4832.9	1.877
	3.8	2299.6	1.817
	7.6	1578.1	1.802
	15.2	1503.8	1.805
F^1	0	27.8	1.974
	1.3	22.7	1.960
	3.8	16.4	1.951
	15.2	14.0	1.938

Table 13. Ratio of Estimated Critical Densities from Eq. (10) to that from Eq. (1) for the Units of Table 1 for R = 15.2 cm.

N	Type of Unit, α					
	A^2	A^4	B^2	C^3	D^2	F^1
27	0.9210	0.9250	0.9298	0.9565	0.9676	0.9776
64	1.0037	1.0187	0.9967	0.9875	0.9936	0.9995
125	1.0341	1.0567	1.0199	0.9919	0.9953	1.0076
216	1.0431	1.0710	1.0254	0.9866	0.9883	1.0101
512	1.0373	1.0717	1.0170	0.9672	0.9671	1.0088
10^3	1.0215	1.0593	1.0006	0.94604	0.9451	1.0046
8×10^3	0.9423	0.9851	0.9233	0.8649	0.8627	0.9821

Since the ratio of N(0)/N(R) as obtained from Eq. (2) does not differ appreciably from that of Eq. (10) we shall utilize the latter because it is more tractable. Using Eq. (10) and maintaining the density as constant, i.e., $\bar{\rho}(\alpha,0) = \bar{\rho}(\alpha,R)$ we find

$$\frac{N(0)}{N(R)} = \frac{A(0)}{A(R)} \quad \frac{\underline{\bar{K}}(\alpha,0)^{s(0)}}{\underline{\bar{K}}(\alpha,R)^{s(R)}} \quad \bar{\rho}^{[s(R) - s(0)]}. \tag{11}$$

Define the quantity $C(\alpha,R)$ as

$$C(\alpha,R) = \frac{A(0)}{A(R)} \quad \frac{\underline{\bar{K}}(\alpha,0)^{s(0)}}{\underline{\bar{K}}(\alpha,R)^{s(R)}} \quad ,$$

and define

$$g(R) \equiv \frac{N(0)}{N(R)} = C(\alpha,R) \quad \bar{\rho}^{-[s(0) - s(R)]} \quad . \tag{12}$$

This relation explicitly displays the dependence of the reflection factor g(R) on density and implicitly, through $C(\alpha,R)$, the dependence on unit size and amount of array reflection. Considering the density dependence, we observe

$$\bar{\rho}^{-\,s'(R)} \begin{array}{l} \geq 1 \text{ for } \rho \leq 1 \\ < 1 \text{ for } \rho > 1 \end{array} ,$$

where $s'(R) = s(0) - s(R)$.

Thus, $g(R) \begin{array}{l} \geq C(\alpha,R) \text{ for } \bar{\rho} \leq 1 \\ < C(\alpha,R) \text{ for } \bar{\rho} > 1. \end{array}$

Alternately stated, for systems with an average uranium density of less than unity, the constant $C(\alpha,R)$ will represent a lower bound to the factor g(R); systems having $\bar{\rho} > 1$ produce an upper bound for g(R). The magnitude of the constant $C(\alpha,R)$ for the case of a 15.2-cm-thick paraffin reflector and its dependence on the unit size and density appear in Table 14. It is emphasized that the actual factor g(R) to be used in applications will be greater than those given if the average density is less

Table 14. Lower Bounds for the Reflection Factors $g(R)$ and $g(\rho)$ as a Function of Unit Size and Density for a Value of R = 15.2 cm.

Unit, α	$C(\alpha,15.2)$, $\rho \leq 1$	$C'(\alpha,15.2)$, N fixed
A^2	17.2	4.2
A^4	15.7	4.1
B^2	20.9	4.7
C^3	21.1	4.8
D^2	24.5	5.3
F^1	6.3	2.6

than unity. As an example, the D^2 units at a $\bar{\rho}$ value of 0.2 g U/cm^3 (20 in. cube) gives $g(R) \sim 27$ which is greater than $C(D^2,15.2) = 24.5$

Also of interest is the case where the number of units is maintained and the addition of a reflector is compensated by a reduction in density $\bar{\rho}$. Again, using Eq. (10) with $N(0) = N(R)$ we obtain

$$\frac{\bar{\rho}(\alpha,0)}{\bar{\rho}(\alpha,R)} = \frac{A(0)^{1/s(0)}}{A(R)^{1/s(R)}} \cdot \frac{\underline{\bar{K}}(\alpha,0)}{\underline{\bar{K}}(\alpha,R)} \, N^{\frac{s(0) - s(R)}{s(0)s(R)}} . \qquad (13)$$

Rewriting this as

$$g(\bar{\rho}) \equiv \frac{\rho(\alpha,0)}{\rho(\alpha,R)} = C'(\alpha,R) \, N^{\frac{s(0) - s(R)}{s(0)s(R)}} \qquad (14)$$

we see that $C'(\alpha,R)$ represents a lower bound for $g(\bar{\rho})$ for all values of $\bar{\rho}$. Values for $C'(\alpha,15.2)$ are also presented in Table 14.

The factors for reflection, $g(R)$ and $g(\bar{\rho})$, are valid for values of N greater than 27 and less than 8000 units. This range is outside that investigated experimentally. One may use the experimental data and Eq. (2), to obtain more accurate values than can be had from a linear extrapolation of the data. Figure 5 gives such a representation for the two units A^2 and C^3; the points are experimental and the lines are from Eq. (2).

REMARKS

The fraction of neutrons leaking from a critical array of metal units, within the statistical uncertainty, is the same as that which leaks from a critical cylinder or sphere of metal, is a constant, and is independent of the size and number of pieces present. Of the two postulates upon which this work is founded, the first, that the neutron leakage fraction from different systems of the same units must be equal, is a special case of the above. The second, an instrument for the utilization of the first, gives an expression for the non-leakage fraction and gives numerical

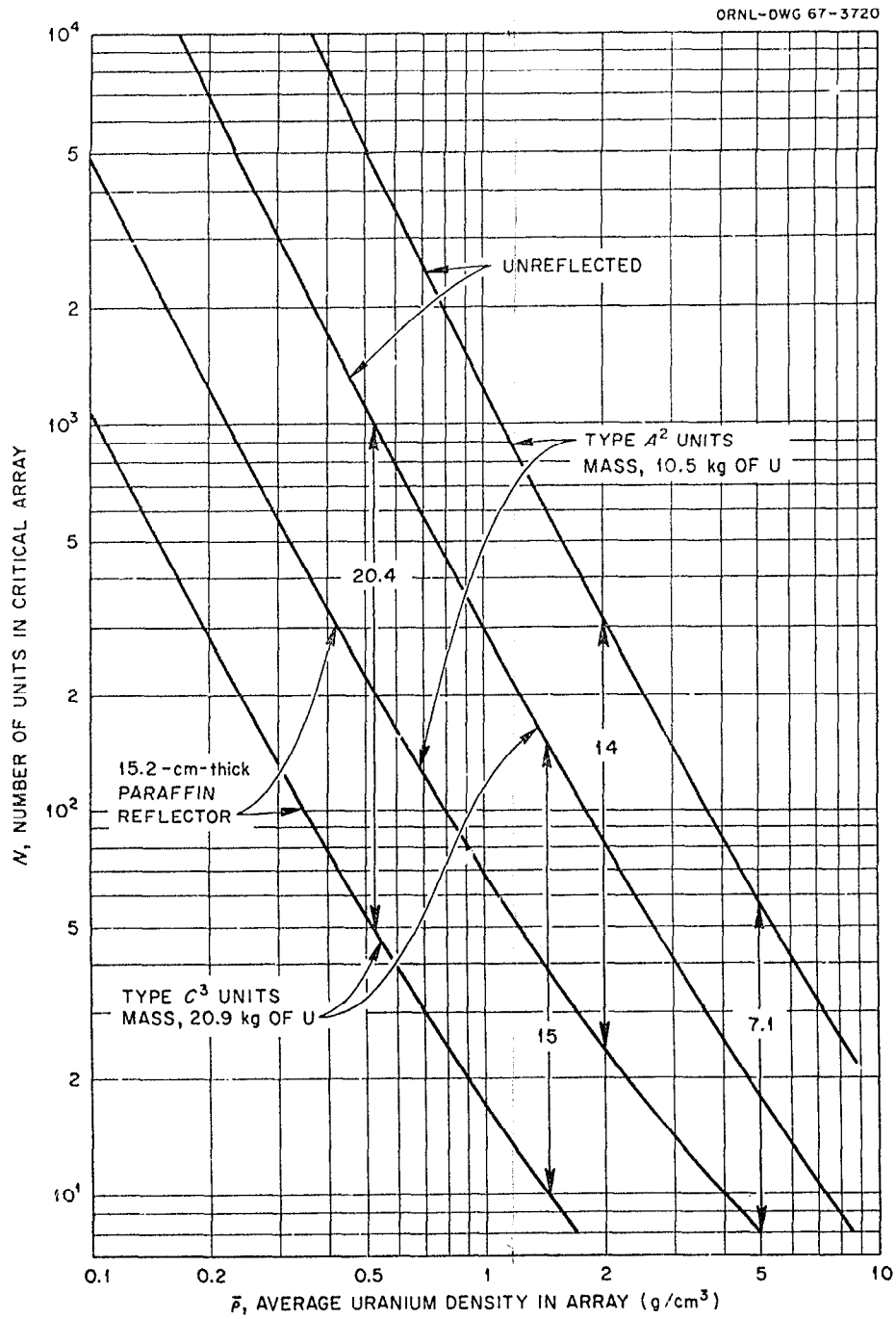

Fig. 5. Reflector Factors Obtained by Comparison of the Critical Dimensions of Unreflected and 15.2-cm-Thick Paraffin Reflected Arrays of Cylindrical U(93) Metal Units.

results quite distinct from the fractions obtained directly from Monte Carlo calculations. Judgement as to the reliability of the two postulates must be based on the results of their application to experimental and non-experimental systems and their valuation through Monte Carlo calculations. Accepting such calculative techniques as a suitable criteria makes plausible the postulates for the purposes delineated herein.

It has been demonstrated that the equation NB_N^2 = constant provides more accurate estimates than its density analogue approximation. The accuracy of the latter, however, can be improved by using the constants A(R) and s(R) characteristic of a given unit rather than the average values derived for the material. The greatest usefulness of the density analogue representation is its ease of application in providing a rapid evaluation which facilitates separating problems into those which have no criticality hazard and those which must be examined with greater detail and accuracy.

The ratio of the number of units in an unreflected critical array to that when a reflector is present has been shown not to be constant but to exhibit a dependence upon unit material and size, average uranium density, and the amount of reflector present. The values for metal units are larger than any previously estimated for such systems in the density range considered here. Any proposed use of a reflection factor for nuclear safety specification should be consistent with the method used to estimate the critical conditions for a system.

The quantity $1 - \underline{\bar{K}}(\alpha,R)$ is similar to the reflection factor utilized in the GEM calculations. Under this guise one can regard $\{1 - \underline{\bar{K}}(\alpha,R)/\underline{\bar{K}}(\alpha,0)\}$ as proportional to the albedo of a reflector about arrays of units. It may be deduced from the tabulated values of $\underline{\bar{K}}(\alpha,R)$ that the fraction of neutrons returned by a thick paraffin reflector ranges from 0.55 to 0.65 for the metal cylinders and is about 0.47 for the F^1 solution units.

The Monte Carlo calculations contained in this paper are for 10^4 neutron histories and result in an accuracy of $\pm$ 0.03 in k at a 95 percent confidence interval. This accuracy was felt to be an adequate yardstick for nuclear safety specification and further expenditure of effort for improvement was not attempted.

ACKNOWLEDGEMENTS

The sustaining interest of Drs. Dixon Callihan and H. C. Paxton has been very beneficial. Their many helpful discussions and those with Mr. D. R. Smith of Los Alamos are acknowledged gratefully. To Messers. G. E. Whitesides and O. W. Hermann of the Computer Sciences Center at K-25 is due credit for the execution of the Monte Carlo calculations.